Linear Transformations
Reading: Lay 1.8

September 13, 2013

One thing that often yields interesting ideas in math is to look at the same object from different points of view. The lecture today is about taking a different perspective on the product $A\mathbf{x}$.

1 Multiplication as a function

We have spent a lot of time on equations of the form

$$A\mathbf{x} = \mathbf{b} \tag{1}$$

Let us turn our attention to the left-hand side of this equation. If $\mathbf{x} \in \mathbb{R}^n$, and if A is an $m \times n$ matrix, then the product $A\mathbf{x}$ is a vector in $\mathbb{R}^m$. If we fix such an $m \times n$ matrix A, then we can view multiplication by A as a function from $\mathbb{R}^n$ to $\mathbb{R}^m$. Recall the definition of a function:

Definition 1.1. Let U and V be two sets. A **function** f from U to V is a rule which assigns to every element x of U exactly one element $f(x) \in V$. U is sometimes called the "**domain**" of f, and V is sometimes called the "**codomain**"; if $x \in U$, we call $f(x)$ the "**image**" of x.

From this point of view, when we solve the equation (1) for some fixed $\mathbf{b}$, what we are really doing is finding the set of all vectors $\mathbf{x} \in \mathbb{R}^n$ which are mapped to $\mathbf{b}$ under the function "multiplication by A."

Another term from the study of functions that we will use is "**range**". The range of a function f is the set of all images $f(x)$ as x is allowed to vary over all elements of the domain. That is, if the domain f is U, then

$$\operatorname{range} f = \{y : y = f(x) \text{ for some } x \in U\}.$$

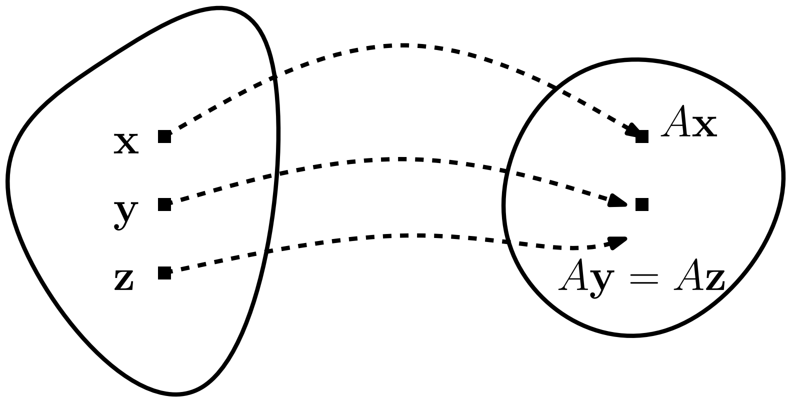

Figure 1: A schematic representation of the function "multiplication by A". The set on the left is the domain, $\mathbb{R}^n$. The set on the right is $\mathbb{R}^m$. Note that here $A\mathbf{y} = A\mathbf{z}$; sometimes this is the case, though not all the time.

Note that in general, the range and the codomain are not equal!

Let B be an $m \times n$ matrix. The function "multiplication by B," as we have defined it, has domain $\mathbb{R}^n$, and its codomain is $\mathbb{R}^m$.

Example 1.2. Given the matrix

$$A = \begin{bmatrix} 1 & 1 \\ 0 & 1 \\ 0 & 0 \end{bmatrix},$$

consider the function defined by multiplication by A. What are its domain, codomain, and range?

A has two columns, so it multiplies vectors in $\mathbb{R}^2$; thus, its domain is $\mathbb{R}^2$. Similarly, its codomain is $\mathbb{R}^3$. Its range is the set of all $\mathbf{y} \in \mathbb{R}^3$ such that $A\mathbf{x} = \mathbf{y}$ for some $\mathbf{x}$ in $\mathbb{R}^2$. That is, the range is equal to

$$\operatorname{span}\left\{ \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}, \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} \right\}. \tag{2}$$

Now, neither of the two columns of A is a multiple of the other. As we know from previous notes (and Lay 1.3), the span of a set of two vectors, when neither vector is a multiple of the other, is a plane in $\mathbb{R}^3$. Thus, the span in (2) is a plane in $\mathbb{R}^3$. Therefore, the range of the multiplication function is not all of the codomain (for instance, no vector with nonzero third coordinate is in this range).

Let us introduce functional notation: start giving names to functions of the form "multiplication by $A\mathbf{x}$". We sometimes write the multiplication-by-A function in symbols as $\mathbf{x} \mapsto A\mathbf{x}$. Sometimes we will also call it by a letter, like T.

Example 1.3. Suppose

$$A = \begin{bmatrix} 1 & 1 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix}, \quad \mathbf{u} = \begin{bmatrix} 1 \\ 1 \\ 3 \end{bmatrix}, \quad \text{and } \mathbf{v} = \begin{bmatrix} -3 \\ -3 \\ 1 \end{bmatrix}.$$

Define the transformation T on $\mathbb{R}^3$ by $T(\mathbf{x}) = A\mathbf{x}$.

1. What is the image of $\mathbf{u}$ under the action of T?
2. Find an $\mathbf{x} \in \mathbb{R}^3$ such that $T(\mathbf{x}) = \mathbf{v}$.
3. Is there more than one possible choice of $\mathbf{x}$ satisfying $T(\mathbf{x}) = \mathbf{v}$?

We list the answers to the questions here:

1. This question is another way of asking what $T(\mathbf{u})$ is equal to. We compute the matrix product and see

$$T(\mathbf{u}) = A\mathbf{u} = 1 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + 1 \begin{bmatrix} 1 \\ 1 \\ 0 \end{bmatrix} + 3 \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} = \begin{bmatrix} 2 \\ 2 \\ 4 \end{bmatrix}.$$

2. This is asking for a solution to the linear system

$$A\mathbf{x} = \mathbf{v}.$$

we write the augmented matrix

$$\begin{bmatrix} 1 & 1 & 0 & -3 \\ 1 & 1 & 0 & -3 \\ 1 & 0 & 1 & 1 \end{bmatrix}. \tag{3}$$

Computing now the RREF of (3), we clear the column below the first pivot:

$$\begin{bmatrix} 1 & 1 & 0 & -3 \\ 0 & 0 & 0 & 0 \\ 0 & -1 & 1 & 4 \end{bmatrix},$$

then interchange the second and third rows:

$$\begin{bmatrix} 1 & 1 & 0 & -3 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix},$$

then clear out the column above the second pivot:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & -1 & 1 & 4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

Lastly, we multiply the second row by -1 to get the RREF:

$$\begin{bmatrix} 1 & 0 & 1 & 1 \\ 0 & 1 & -1 & -4 \\ 0 & 0 & 0 & 0 \end{bmatrix}.$$

This corresponds to a linear system with one free variable x_3. The equations are

$$x_1 + x_3 = 1 \tag{4}$$
$$x_2 - x_3 = -4. \tag{5}$$

One solution to this system is given by $x_1 = 0, x_2 = -3, x_3 = 1$. This corresponds to

$$\mathbf{x} = \begin{bmatrix} 0 \\ -3 \\ 1 \end{bmatrix}.$$

3. As stated in item 2 just above, the linear system we just calculated has a free variable. So there are infinitely many choices of $T(\mathbf{x}) = \mathbf{v}$. For instance, choosing $x_3 = 2$ in (4) and (5) gives the solution

$$\mathbf{x} = \begin{bmatrix} -1 \\ -2 \\ 2 \end{bmatrix}.$$

Example 1.4. Consider

$$A = \begin{bmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \mathbf{v} = \begin{bmatrix} 1 \\ 2 \\ 5 \end{bmatrix}$$

and let T be the function defined by $T(\mathbf{x}) = A\mathbf{x}$.

1. Is there a vector $\mathbf{x}$ such that $T(\mathbf{x}) = \mathbf{v}$?
2. Is there more than one?

Write the augmented matrix:

$$\begin{bmatrix} 0 & -1 & 0 & 1 \\ 1 & 0 & 0 & 2 \\ 0 & 0 & 1 & 5 \end{bmatrix}.$$

The RREF of this augmented matrix is:

$$\begin{bmatrix} 1 & 0 & 0 & 2 \\ 0 & 1 & 0 & -1 \\ 0 & 0 & 1 & 5 \end{bmatrix}. \tag{6}$$

Thus, $T(\mathbf{x}) = \mathbf{v}$ is solved by

$$\mathbf{x} = \begin{bmatrix} 2 \\ -1 \\ 5 \end{bmatrix}.$$

There is no free variable in the system (6), so there is no other solution.

(The T in this example can be described geometrically as the function which takes a vector in $\mathbb{R}^3$ and rotates it by 90° around the z axis. Think about this geometric description: do our answers to the questions about $T(\mathbf{x}) = \mathbf{v}$ make sense?)

2 Linear Transformations

Definition 2.1. A linear transformation is a mapping T (that is, a function) from $\mathbb{R}^n$ to $\mathbb{R}^m$ (for some n and m) with the following two properties:

1. $T(\mathbf{u}+\mathbf{v}) = T(\mathbf{u}) + T(\mathbf{v})$ for all $\mathbf{u}, \mathbf{v}$ in the domain of T;

2. $T(c\mathbf{u}) = cT(\mathbf{u})$ for all scalars c and all $\mathbf{u}$ in the domain of T.

Note every mapping of the form $\mathbf{x} \mapsto A\mathbf{x}$ is a linear transformation. The definitions of a linear transformation immediately imply the two following things (left for you as an easy exercise):

1. For any linear transformation T, we have $T(\mathbf{0}) = \mathbf{0}$ (note these two zero vectors may not be the same–may have different numbers of components)

2. For any vectors $\mathbf{v}_1, \ldots, \mathbf{v}_p$ in the domain of T, and any scalars $c_1 \ldots c_p$, we have

$$T(c_1\mathbf{v}_1 + \ldots + c_p\mathbf{v}_p) = c_1T(\mathbf{v}_1) + \ldots + c_pT(\mathbf{v}_p).$$